On maxima of random walks in varying environments

Joint work with Hongyan SUN

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目　录

Question we are concerned

X_n : a random walk on $\mathbb{Z}^+$, either nearest-neighbor or non-nearest-neighbor, starting from X_0. Let

$$D = \inf\{n \geq 0 : X_n < X_0\}$$

be the first 'return' time and

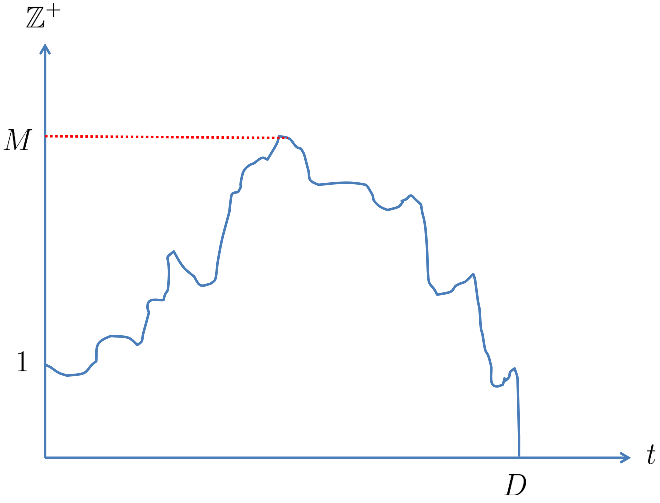
$$M = \max\{X_n : 0 \leq n \leq D\}$$

be the maximum of the excursion $\{X_0, X_1, ..., X_D\}$.

Questions:

- ♠ Compute $P(M = n, D < \infty) = ?$ **(easy task!)**
- ♠ Find $f(n)$ such that

$$\lim_{n\to\infty} f(n)P(M = n, D < \infty) = c > 0.$$ **(not so easy!!)**



$\mathbb{Z}^+$
M
1
t
D

Basic points

We study three models:

- Nearest-neighbor random walk X.
- (2,1) random walk Y.
- (1,2) random walk Y'.

♠ Nearest-neighbor setting $\leftrightsquigarrow$ product of a sequence of numbers.

♠ (2,1) random walk $\leftrightsquigarrow$ product of nonhomogeneous matrices.

♠ (1,2) random walk $\leftrightsquigarrow$ product of a sequence of continued fractions.

Nearest Setting

Suppose $\{X_n\}$ is a nearest neighbor random walk:

$$\begin{aligned}
&P(X_0 = 1) = 1, \\
&P(X_n = 1 | X_{n-1} = 0) = 1, \\
&P(X_n = k+1 | X_{n-1} = k) = p_k \\
&P(X_n = k-1 | X_{n-1} = k) = q_k := 1 - p_k, k \geq 1,
\end{aligned}$$

where $p_k \in (0,1), \forall k \geq 1$. Write $\rho_k = \frac{q_k}{p_k}$, and define for $a < k < b$,

$$P_k(a, b, -) = P(X \text{ hits } a \text{ before } b | X_0 = k).$$

Lemma

For $0 \leq a \leq k \leq b$, we have

$$P_k(a, b, -) = \frac{\sum_{j=k}^{b-1} \rho_{a+1} \cdots \rho_j}{1 + \sum_{j=a+1}^{b-1} \rho_{a+1} \cdots \rho_j}.$$

Corollary

For the chain $\{X_n\}$, we have

$$P(M = n, D < \infty) = (1 - P_1(0, n, -))P_n(0, n+1, -)$$
$$= \frac{1}{1 + \sum_{j=1}^{n-1} \rho_1 \cdots \rho_j} \cdot \frac{\rho_1 \cdots \rho_n}{1 + \sum_{j=1}^{n} \rho_1 \cdots \rho_j}.$$

Simple random walk

Suppose that $p_i \equiv p \in (0,1), i \geq 1$ and let $\rho := \frac{1-p}{p}$. Then

$$P(M = n, D < \infty) \sim \begin{cases} \frac{1}{n(n+1)}, & \rho = 1, \\ (1-\rho)^2 \rho^n, & \rho < 1, \\ (1-\rho)^2 \rho^{-(n+1)}, & \rho > 1, \end{cases} \quad \text{as } n \to \infty.$$

Thus,

(a) if $\rho = 1$, X is *null recurrent* and $P(M = n)$ decays *polynomially*;

(b) if $\rho < 1$, X is *transient* and $P(M = n, D < \infty)$ decays *exponentially*;

(c) if $\rho > 1$, X is *positive recurrent* and $P(M = n)$ decays *exponentially*.

Question: Besides the polynomial and exponential ones, **can $P(M=n, D<\infty)$ decays with other rates?**

YES: adding some perturbation on the recurrent simple random walk.

For $K \geq 1,\ B \in \mathbb{R}$, set

$$\Lambda(1,i,B) = \frac{B}{i}, \Lambda(2,i,B) = \frac{1}{i} + \frac{B}{i \log i}, \dots,$$

$$\Lambda(K,i,B) = \frac{1}{i} + \frac{1}{i \log i} + \dots + \frac{1}{i \log i \cdots \log_{K-2} i} + \frac{B}{i \log i \cdots \log_{K-1} i},$$

where $\log_0 i = i, \log_1 i = \log i, .., \log_K i = \log \log_{K-1} i$. Set

$$i_0 = \min\left\{ i : \log_{K-1} i > 0, \frac{|\Lambda(K,i,B)|}{4} < \frac{1}{2} \right\}.$$

For fixed $B \in \mathbb{R}$ and $K = 1, 2, \dots$ set

$$r_i = \begin{cases} \frac{\Lambda(K,i,B)}{4}, & i \geq i_0, \\ r_{i_0}, & i < i_0. \end{cases}$$

Theorem 1

Fix $K \geq 1$ and $B \in \mathbb{R}$.

(i) If $p_i = \frac{1}{2} + r_i, i \geq 1$, then, as $n \to \infty$,

$$P(M = n, D < \infty) \sim \begin{cases} \frac{c}{n \log n \log\log n \cdots \log_{K-1} n (\log_K n)^2}, & \text{if } B = 1, \\ \frac{c}{n \log n \log\log n \cdots \log_{K-2} n (\log_{K-1} n)^B}, & \text{if } B > 1, \\ \frac{c}{n \log n \log\log n \cdots \log_{K-2} n (\log_{K-1} n)^{2-B}}, & \text{if } B < 1. \end{cases}$$

(ii) If $p_i = \frac{1}{2} - r_i, i \geq 1$, then, as $n \to \infty$,

$$P(M = n, D < \infty) \sim \begin{cases} \frac{c}{n^{B+2}}, & \text{if } K = 1, B > -1, \\ \frac{c}{n (\log n)^2}, & \text{if } K = 1, B = -1, \\ c n^B, & \text{if } K = 1, B < -1, \\ \frac{c}{n^3 \log n \ldots \log_{K-2} n (\log_{K-1}^n)^B}, & \text{if } K > 1. \end{cases}$$

Proof sketch: We already knew that

$$P(M = n, D < \infty) = \frac{1}{1 + \sum_{j=1}^{n-1} \rho_1 \cdots \rho_j} \cdot \frac{\rho_1 \cdots \rho_n}{1 + \sum_{j=1}^{n} \rho_1 \cdots \rho_j}.$$

By some delicate computation, we can show

Lemma. Fix $K = 1, 2, ..$ and $B \in \mathbb{R}$.

(a) If $p_i = \frac{1}{2} + r_i, i \geq 1$, then

$$\rho_1 \cdots \rho_n \sim \frac{c}{n \log n \log\log n \cdots \log_{K-2} n (\log_{K-1} n)^B}, \text{as } n \to \infty.$$

(b) If $p_i = \frac{1}{2} - r_i, i \geq 1$, then

$$\rho_1 \cdots \rho_n \sim c n \log n \log\log n \cdots \log_{K-2} n (\log_{K-1} n)^B, \text{as } n \to \infty.$$

Furthermore, though it is very complicated, the term $\sum_{j=1}^{n} \rho_1 \cdots \rho_j$ can be estimated by using the above lemma. □

Suppose

$$\forall k \geq 2, q_k, p_k > 0,\ q_k + p_k = 1.$$

(2,1) Random walk Y

$Y = \{Y_k\}_{k\geq 0}$: a Markov chain on $\mathbb{Z}_+$ such that

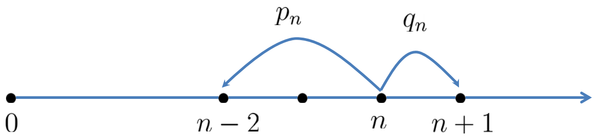
$$P(Y_{k+1} = 1|Y_k = 0) = P(Y_{k+1} = 2|Y_k = 1) = 1,$$
$$P(Y_{k+1} = n+1|Y_k = n) = q_n,$$
$$P(Y_{k+1} = n-2|Y_k = n) = p_n, n \geq 2, k \geq 0.$$

(1,2) Random walk Y'

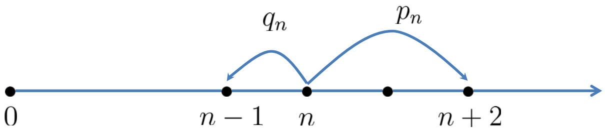
$Y' = \{Y'_k\}_{k\geq 0}$: a Markov chain on $\mathbb{Z}_+$ such that

$$P(Y'_{k+1} = 0|Y'_k = 1) = P(Y'_{k+1} = 2|Y'_k = 0) = 1,$$
$$P(Y'_{k+1} = n-1|Y'_k = n) = q_n,$$
$$P(Y'_{k+1} = n+2|Y'_k = n) = p_n, n \geq 2, k \geq 0.$$

- Unless otherwise stated, we always assume that both Y and Y' start from $y_0 = y_0' = 2$.
- Y' is usually called the **adjoint chain** of Y and vice versa.



(2,1) random walk Y



(1,2) random walk Y'

For $k \geq 2$, introduce matrix

$$N_k := \begin{pmatrix} \theta_k & \theta_k \\ 1 & 0 \end{pmatrix} \text{ with } \theta_k := \frac{p_k}{q_k}.$$

Proposition 1

Consider **(2,1) random walk** Y. For $n \geq 2$, we have

$$P(M = n, D < \infty) = \frac{1}{1 + \sum_{s=2}^{n-1} \mathbf{e}_1 N_s \cdots N_2 \mathbf{e}_1^t} \frac{\mathbf{e}_1 N_n \cdots N_2 \mathbf{e}_1^t}{1 + \sum_{s=2}^{n} \mathbf{e}_1 N_s \cdots N_2 \mathbf{e}_1^t}.$$

We see from Proposition 1 that, in order to study the limit behavior of $P(M = n, D < \infty)$, one has to study at first the asymptotics of

$$\mathbf{e}_1 N_n \cdots N_2 \mathbf{e}_1^t$$

which involve the asymptotics of the **product of nonhomogeneous matrices** and are extremely complicated.

We expect that

$$\mathbf{e}_1 N_n \cdots N_2 \mathbf{e}_1^t \sim c\varrho(N_n) \cdots \varrho(N_2)$$

with $\varrho(N_k)$ the **spectral radius** of N_k.

Let a_k, b_k, d_k, $k \geq 1$ be certain positive numbers and for $k \geq 1$, set

$$A_k = \begin{pmatrix} a_k & b_k \\ d_k & 0 \end{pmatrix}. \tag{1}$$

(B1) For some $\sigma > 0$, $a_k, b_k, d_k \geq \sigma$ for all $k \geq 1$ and

$$\sum_{k=2}^{\infty} |a_k - a_{k-1}| + |b_k - b_{k-1}| + |d_k - d_{k-1}| < \infty.$$

Under (B1),

$$A_k = \begin{pmatrix} a_k & b_k \\ d_k & 0 \end{pmatrix} \to \begin{pmatrix} a & b \\ d & 0 \end{pmatrix} =: A \tag{2}$$

for proper $a, b, d > 0$.

Suppose now condition (B1) holds. We introduce further the following conditions **which are mutually exclusive**.

(B2)$_\mathbf{a}$ $\exists k_0 > 0$, such that $\frac{a_k}{b_k} = \frac{a_{k+1}}{b_{k+1}},\ \frac{d_k}{b_k} \neq \frac{d_{k+1}}{b_{k+1}},\ \forall k \geq k_0$ and

$$\lim_{k\to\infty} \frac{d_k/b_k - d_{k+1}/b_{k+1}}{d_{k+1}/b_{k+1} - d_{k+2}/b_{k+2}}$$

exists as a finite or infinite number.

(B2)$_\mathbf{b}$ $\exists k_0 > 0$, such that $\frac{a_k}{b_k} \neq \frac{a_{k+1}}{b_{k+1}},\ \frac{d_k}{b_k} = \frac{d_{k+1}}{b_{k+1}},\ \forall k \geq k_0$ and

$$\lim_{k\to\infty} \frac{a_k/b_k - a_{k+1}/b_{k+1}}{a_{k+1}/b_{k+1} - a_{k+2}/b_{k+2}}$$

exists as a finite or infinite number.

(B2)$_\mathbf{c}$ $\exists k_0 > 0$, such that $\frac{a_k}{b_k} \neq \frac{a_{k+1}}{b_{k+1}}$, $\frac{d_k}{b_k} \neq \frac{d_{k+1}}{b_{k+1}}$, $\forall k \geq k_0$ and

$$\tau := \lim_{k\to\infty} \frac{d_k/b_k - d_{k+1}/b_{k+1}}{a_k/b_k - a_{k+1}/b_{k+1}} \neq \frac{-a+\sqrt{a^2+4bd}}{2b}$$

exists as a finite or infinite number. In addition, if τ is finite, assume further $\lim_{k\to\infty} \frac{a_k/b_k - a_{k+1}/b_{k+1}}{a_{k+1}/b_{k+1} - a_{k+2}/b_{k+2}}$ exists as a finite or infinite number. Otherwise, if $\tau = \infty$, assume further $\lim_{k\to\infty} \frac{d_k/b_k - d_{k+1}/b_{k+1}}{d_{k+1}/b_{k+1} - d_{k+2}/b_{k+2}}$ exists as a finite or infinite number.

Remark.

- ♠ Conditions **(B2)$_\mathbf{a}$**, **(B2)$_\mathbf{b}$** and **(B2)$_\mathbf{c}$** look very awkward, but it is easy to find examples such that one of them is satisfied.
- ♠ Roughly speaking, it requires that a_k, b_k and d_k may fluctuate in different orders, but should fluctuate in some common manner.

Theorem 2

Suppose condition (B1) and one of $(\mathrm{B2})_a$, $(\mathrm{B2})_b$ and $(\mathrm{B2})_c$ hold. Then $\forall i, j \in \{1,2\}$, there exists $0 < c < \infty$ such that

$$\lim_{k\to\infty} \frac{\mathbf{e}_i A_k \cdots A_1 \mathbf{e}_j^t}{\varrho(A_k)\cdots\varrho(A_1)} = c.$$

Remark. Theorem 2 has been generalized to general nonnegative matrices

$$A_k = \begin{pmatrix} a_k & b_k \\ d_k & \theta_k \end{pmatrix}.$$

H. M. Wang. On extinction time distribution of a 2-type linear-fractional branching process in varying environment with asymptotically constant mean matrices. *arXiv:* 2106.01203, 2021.

Sketched proof of Theorem 2.

Step 1. Show that $c_3 < \frac{\mathbf{e}_1 A_k \cdots A_1 \mathbf{e}_1^t}{\varrho(A_k)\cdots\varrho(A_1)} < c_4$.

(1) Show that $\varrho(A_k)\cdots\varrho(A_1) \asymp \varrho(A_k \cdots A_1)$.

(2) Show that $\varrho(A_k \cdots A_1) \sim \phi \mathbf{e}_1 A_k \cdots A_1 \mathbf{e}_1^t$.

Step 2. Show that $\lim_{m\to\infty} \frac{\mathbf{e}_1 A_k \cdots A_m \mathbf{e}_1^t}{\varrho(A_k)\cdots\varrho(A_m)} = c.$

Set $x_k := \frac{\mathbf{e}_1 A_k \cdots A_1 \mathbf{e}_1^t}{\varrho(A_k)\cdots\varrho(A_1)}$ and $f_k := \frac{\mathbf{e}_2 A_k \cdots A_1 \mathbf{e}_1^t}{\mathbf{e}_1 A_k \cdots A_1 \mathbf{e}_1^t}$. Then

$$
\begin{aligned}
&c_3 < x_k < c_4,\\
&x_{k+1} - x_k = (\varrho(A_{k+1})x_k)^{-1}(a_{k+1} - \varrho(A_{k+1}) + b_{k+1}f_k),\\
&\frac{x_{k+1}}{x_k} = \frac{1}{\varrho(A_{k+1})}\left(a_{k+1} + b_{k+1}f_k\right),\\
&f_k = \frac{\beta_k}{\alpha_k +} \frac{\beta_{k-1}}{\alpha_{k-1} +} \cdots \frac{\beta_2}{\alpha_2 +} \frac{\beta_1}{\alpha_1}, \text{ with } \alpha_k = \frac{a_k}{b_k}, \beta_k = \frac{d_k}{b_k}.
\end{aligned}
$$

f_k is usually referred to as the **critical tail** of a continued fraction.

Critical tail sequence of a continued fraction

Lemma

Suppose that $\alpha_k, \beta_k, \omega_k > 0, k \geq 1$ are numbers such that $\alpha_k \to \alpha$, $\beta_k \to \beta$, and $\omega_k \to \frac{\sqrt{\alpha^2+4\beta}-\alpha}{2} =: \omega$, as $k \to \infty$, where $0 < \alpha, \beta < \infty$ are certain constants. For $k \geq 1$, let

$$f_k := \frac{\beta_k}{\alpha_k} + \frac{\beta_{k-1}}{\alpha_{k-1}} + \cdots \frac{\beta_2}{\alpha_2} + \frac{\beta_1}{\alpha_1}.$$

Set

$$\varepsilon_k = f_k - \omega_k, k \geq 1 \text{ and } \delta_k = \beta_k - \omega_k(\alpha_k + \omega_{k-1}), k \geq 2.$$

Let q be a fixed number. We have

$$\text{if } \lim_{k\to\infty} \frac{\varepsilon_k}{\varepsilon_{k+1}} = q, \text{ then } |q| \geq 1 \text{ and } \lim_{k\to\infty} \frac{\delta_k}{\delta_{k+1}} = q;$$

$$\text{if } \lim_{k\to\infty} \frac{\delta_k}{\delta_{k+1}} = q, \text{ then } |q| \geq 1 \text{ and } \lim_{k\to\infty} \frac{\varepsilon_k}{\varepsilon_{k+1}} = q \text{ or } -\frac{1+\omega}{\omega}.$$

.

Lemma

Suppose condition (B1) and one of $(B2)_a$, $(B2)_b$ and $(B2)_c$ hold. Set $\beta_k = d_k/b_k, \alpha_k = a_k/b_k, k \geq 1$. Let $\omega_k := \frac{\sqrt{\alpha_{k+1}^2+4\beta_{k+1}}-\alpha_{k+1}}{2}, k \geq 1$ and $\delta_k = \beta_k - \omega_k(\alpha_k + \omega_{k-1}), k \geq 2$. Then $\lim_{k\to\infty} \frac{\delta_k}{\delta_{k+1}}$ exists as an finite or infinite number.

Using the above two lemmas, we can show that one of the following three cases happens:

(i) $\sum_{m\geq 2} |x_m - x_{m-1}| < \infty$;

(ii) $x_{m+1} - x_m, m > N_1$ converges to 0 alternatively;

(iii) $x_m, m > N_2$ is monotone in m.

Consequently $\lim_{k\to\infty} x_k = \lim_{k\to\infty} \frac{\mathbf{e}_1 A_k \cdots A_1 \mathbf{e}_1^t}{\varrho(A_k)\cdots\varrho(A_1)} = c.$

Theorem 2 is proven. $\square$

Maximum of (2,1) random walk

Let
$$r_i := \begin{cases} \frac{\Lambda(K,i,B)}{3}, & i \geq i_0, \\ r_{i_0}, & i < i_0, \end{cases}$$

Theorem 3

Consider **(2,1) random walk** Y. Fix $K = 1, 2, 3, \ldots$ and $B \in \mathbb{R}$.

(i) If $q_i = \frac{2}{3} + r_i, i \geq 2,$ then, as $n \to \infty$,

$$P(M = n, D < \infty) \sim \begin{cases} \frac{c}{n \log n \cdots \log_{K-2} n \log_{K-1} n (\log_K n)^2}, & \text{if } B = 1, \\ \frac{c}{n \log n \cdots \log_{K-2} n (\log_{K-1} n)^B}, & \text{if } B > 1, \\ \frac{c}{n \log n \cdots \log_{K-2} n (\log_{K-1} n)^{2-B}}, & \text{if } B < 1. \end{cases}$$

(ii) If $q_i = \frac{2}{3} - r_i, i \geq 2,$ then, as $n \to \infty$,

$$P(M = n, D < \infty) \sim \begin{cases} \frac{c}{n^{B+2}}, & \text{if } K = 1, B > -1, \\ \frac{c}{n (\log n)^2}, & \text{if } K = 1, B = -1, \\ c n^B, & \text{if } K = 1, B < -1, \\ \frac{c}{n^3 \log n \ldots \log_{K-2} n (\log_{K-1} n)^B}, & \text{if } K > 1. \end{cases}$$

Sketched proof of Theorem 3.

Note that

$$\varrho(N_k) = \left(\theta_k + \sqrt{\theta_k^2 + 4\theta_k}\right)/2.$$

If $q_i = \frac{2}{3} \pm r_i, i \geq 2,$ then by Taylor enpension of $\varrho(N_k)$ at 0, we get

$$\varrho(N_k) = 1 \mp 3r_k + O(r_k^2) \text{ as } k \to \infty. \tag{3}$$

The proposition below yields the asymptotics of $\varrho(N_k) \cdots \varrho(N_1)$.

Proposition 2

Suppose that $\sigma_i, i \geq 2$ is a sequence of numbers such that

$$\sigma_i = 1 \pm 3r_i + O(r_i^2) \text{ as } i \to \infty.$$

Then we have as $n \to \infty$,

$$\sigma_2 \cdots \sigma_n \sim c\left(n \log n \cdots \log_{K-2} n (\log_{K-1} n)^B\right)^{\pm 1},$$

$$\frac{\sigma_2 \cdots \sigma_n}{\sum_{i=1}^n \sigma_2 \cdots \sigma_i} \to 0.$$

Recall that $N_k := \begin{pmatrix} \theta_k & \theta_k \\ 1 & 0 \end{pmatrix}$ with $\theta_k := \frac{p_k}{q_k}$.

By the following lemma, for the product $N_k \cdots N_2, k \geq 2$, requirements of Theorem 2 are all fulfilled.

Lemma

Whenever $q_i = 2/3 \pm r_i, i \geq 2$,

(i) we have $\lim\limits_{n\to\infty} \frac{r_n - r_{n+1}}{n^2} = 1/3$ and thus $\sum_{k=2}^{\infty} |\theta_{k+1} - \theta_k| < \infty$ **(B1)**;

(ii) for $k \geq i_0$, we have $\frac{1}{\theta_k} \neq \frac{1}{\theta_{k+1}}$ and $\lim_{k\to\infty} \frac{\theta_{k+1}-\theta_k}{\theta_{k+2}-\theta_{k+1}} = 1$ **(B2)**.

Applying Theorem 2 and Proposition 2, if $q_i = \frac{2}{3} \pm r_i, i \geq 2,$ we get as $n \to \infty$,

$$\mathbf{e}_1 N_n \cdots N_2 \mathbf{e}_1^t \sim c\varrho(N_2)\cdots\varrho(N_n)$$
$$\sim c\left(n \log n \cdots \log_{K-2} n (\log_{K-1} n)^B\right)^{\mp 1}.$$

But by Proposition 1, for (2,1) random walk Y, we have

$$P(M=n, D<\infty) = \frac{1}{1+\sum_{s=2}^{n-1} \mathbf{e}_1 N_s \cdots N_2 \mathbf{e}_1^t} \frac{\mathbf{e}_1 N_n \cdots N_2 \mathbf{e}_1^t}{1+\sum_{s=2}^{n} \mathbf{e}_1 N_s \cdots N_2 \mathbf{e}_1^t}.$$

Consequently, Theorem 3 can be proved by an argument similar to the proof Theorem 1(**Nearest-neighbor setting**). □

Finally, we consider **(1,2) random walk** Y' which is more difficult. To derive similar result, besides the asymptotics of product of nonnegative matrices, we need to develop further some other techniques related to the limit periodic continued fractions and the hitting probabilities of the walk.

Maximum of (1,2) random walk

Theorem 4

Consider **(1,2) random walk** Y'. Fix $K = 1, 2, 3, \ldots$ and $B \in \mathbb{R}$.

(i) If $p_i = \frac{1}{3} + r_i, i \geq 2$, then, as $n \to \infty$,

$$P(M = n, D < \infty) \sim \begin{cases} \frac{c}{n \log n \cdots \log_{K-2} n \log_{K-1} n (\log_K n)^2}, & \text{if } B = 1, \\ \frac{c}{n \log n \cdots \log_{K-2} n (\log_{K-1} n)^B}, & \text{if } B > 1, \\ \frac{c}{n \log n \cdots \log_{K-2} n (\log_{K-1} n)^{2-B}}, & \text{if } B < 1. \end{cases}$$

(ii) If $p_i = \frac{1}{3} - r_i, i \geq 2$, then, as $n \to \infty$,

$$P(M = n, D < \infty) \sim \begin{cases} \frac{c}{n^{B+2}}, & \text{if } K = 1, B > -1, \\ \frac{c}{n(\log n)^2}, & \text{if } K = 1, B = -1, \\ cn^B, & \text{if } K = 1, B < -1, \\ \frac{c}{n^3 \log n \ldots \log_{K-2} n (\log_{K-1} n)^B}, & \text{if } K > 1. \end{cases}$$

What is the difficulty?

Let

$$\mathcal{P}_k(m,n,+) = P(Y' \text{ hits } [n,\infty] \text{ before } [0,m] | Y_0' = k)$$
$$\mathcal{P}_k^n(m,n,+) = P(Y' \text{ hits } [n,\infty] \text{ at } n \text{ before } [0,m] | Y_0' = k),$$
$$\mathcal{P}_k^{n+1}(m,n,+) = P(Y' \text{ hits } [n,\infty] \text{ at } n+1 \text{ before } [0,m] | Y_0' = k).$$

Clearly, $\mathcal{P}_k(m,n,+) = \mathcal{P}_k^n(m,n,+) + \mathcal{P}_k^{n+1}(m,n,+)$.

By Markov property, we can get

$$P(M = n, D < \infty) = \mathcal{P}_2^n(1,n,+)(1 - \mathcal{P}_n(1,n+1,+)).$$

It can be shown that

$$1 - \mathcal{P}_n(1,n+1,+) = \frac{1}{1 + \sum_{s=2}^n \mathbf{e}_1 N_s \cdots N_n \mathbf{e}_1^t},$$

$$\mathcal{P}_2^n(1,n,+) = \mathbf{e}_1 N_2 \cdots N_{n-1} \Big(\frac{1 + \sum_{s=2}^{n-1} \mathbf{e}_1 N_s \cdots N_{n-1} \mathbf{e}_2^t}{1 + \sum_{s=2}^{n-1} \mathbf{e}_1 N_s \cdots N_{n-1} \mathbf{e}_1^t} \mathbf{e}_1^t - \mathbf{e}_2^t \Big),$$

which are hard to estimate even though we know that $\mathbf{e}_1 N_n \cdots N_s \mathbf{e}_1^t \sim c(s)\varrho(N_n) \cdots \varrho(N_s)$, since every summand there depends on n.

Continued fraction and escape probability

For $n \geq 2$, set

$$\xi_n \equiv \frac{\theta_n^{-1}}{1} + \frac{\theta_{n+1}^{-1}}{1} + \frac{\theta_{n+2}^{-1}}{1} + \cdots$$

The next lemma gives the escape probabilities of (1,2) random walk.

Lemma(Letchikov 1988)

$$\frac{\xi_2 \cdots \xi_n}{1 + \sum_{i=2}^{n} \xi_2 \cdots \xi_i} \leq 1 - \mathcal{P}_n(1, n+1, +) \leq \frac{\xi_2 \cdots \xi_n + \xi_2 \cdots \xi_{n+1}}{1 + \sum_{i=2}^{n+1} \xi_2 \cdots \xi_i},$$

$$\frac{1}{1 + \sum_{i=2}^{n} \xi_2 \cdots \xi_i} \leq \mathcal{P}_2(1, n, +) \leq \frac{1}{1 + \sum_{i=2}^{n-1} \xi_2 \cdots \xi_i}.$$

We note that

(i) the upper bound of the term $1 - \mathcal{P}_n(1, n+1, +)$ is approximately twice as much as the lower bound, so that it is not enough for us to get the accurate limit behavior of $1 - \mathcal{P}_n(1, n+1, +)$;

(ii) what we need indeed is not $\mathcal{P}_2(1, n, +)$ but $\mathcal{P}_2^n(1, n, +)$.

But we know that

$$\mathcal{P}_2(1, n, +) = \mathcal{P}_2^n(1, n, +) + \mathcal{P}_2^{n+1}(1, n, +).$$

Lemma

Suppose that $p_i = \frac{1}{3} \pm r_i, i \geq 2$. Then we have

$$\lim_{n \to \infty} \frac{\mathcal{P}_2^n(1, n, +)}{\mathcal{P}_2^{n+1}(1, n, +)} = 2.$$

The proof of the lemma is **a long journey.**

The idea is to **construct a new Markov chain related to** Y'.

Let

$$E_n = \{Y' \text{ hits } [n, \infty) \text{ before it hits } [0, 1]\}.$$

Define a measure $\tilde{P}$ by

$$\tilde{P}(\cdot) = P(\cdot | E_n).$$

Let

$$T_n := \inf\{k \geq 0 : Y'_k \in [n, \infty)\}, n \geq 3.$$

Then $T_n < \infty$ almost surely.

We can show that Y' is a Markov chain under $\tilde{P}$ with transition probabilities

$$\tilde{P}(Y'_{k+1} = 4 | Y'_k = 2, k < T_n) = 1,$$

$$\tilde{P}(Y'_{k+1} = i+2 | Y'_k = i, k < T_n) = p_i \frac{\mathcal{P}_{i+2}(1, n, +)}{\mathcal{P}_i(1, n, +)} =: \tilde{p}_i,$$

$$\tilde{P}(Y'_{k+1} = i-1 | Y'_k = i, k < T_n) = 1 - \tilde{p}_i =: \tilde{q}_i, 3 \leq i \leq n-1.$$

Based on this fact, the lemma can be proved by some delicate analysis of the hitting times of the new Morkov chain.

Finally, we deal with the term

$$1 - \mathcal{P}_n(1, n+1, +) = \frac{1}{1 + \sum_{s=2}^{n} \mathbf{e}_1 N_s \cdots N_n \mathbf{e}_1^t}.$$

Lemma

If $p_i = 1/3 \pm r_i, i \geq 2$, then

$$1 - \mathcal{P}_n(1, n+1, +) \sim c \frac{\xi_2 \cdots \xi_n}{\sum_{s=2}^{n+1} \xi_2 \cdots \xi_{s-1}}, \text{ as } n \to \infty.$$

Idea of proof. For $2 \leq s \leq n+1$, set

$$y_{s,n} := \mathbf{e}_1 N_s \cdots N_n \mathbf{e}_1^t \text{ and } \xi_{s,n} := \frac{y_{s+1,n}}{y_{s,n}}.$$

Then $$\mathbf{e}_1 N_s \cdots N_n \mathbf{e}_1^t = y_{s,n} = \xi_{s,n}^{-1} \cdots \xi_{n,n}^{-1}.$$

Thus we obtain

$$1-\mathcal{P}_n(1,n+1,+)=\frac{1}{\sum_{s=2}^{n+1}\xi_{s,n}^{-1}\cdots\xi_{n,n}^{-1}}=\frac{\xi_{2,n}\cdots\xi_{n,n}}{\sum_{s=2}^{n+1}\xi_{2,n}\cdots\xi_{s-1,n}}.$$

If we can show

$$\xi_{2,n}\cdots\xi_{n,n}\sim c\xi_2\cdots\xi_n, \tag{4}$$

$$\sum_{s=2}^{n+1}\xi_{2,n}\cdots\xi_{s-1,n}\sim\sum_{s=2}^{n+1}\xi_2\cdots\xi_{s-1}, \tag{5}$$

as $n\to\infty$, then

$$1-\mathcal{P}_n(1,n+1,+)\sim c\frac{\xi_2\cdots\xi_n}{\sum_{s=2}^{n+1}\xi_2\cdots\xi_{s-1}}.$$

It can be shown that

$$\xi_{s,n}=\frac{\theta_s^{-1}}{1}+\frac{\theta_{s+1}^{-1}}{1}+\cdots+\frac{\theta_n^{-1}}{1}.$$

Then (4) and (5) can be proved with the help of the limit theory of limit periodic continued fraction and Theorem 2($\mathbf{e}_1 N_n \cdots N_2 \mathbf{e}_1^t \sim c\varrho(N_2)\cdots\varrho(N_n)$). □

The lemma below presents several limit behaviors related to $\xi_n, n \geq 2$.

Lemma(W. 2019)

If $p_i = 1/3 \pm r_i, i \geq 2$, then we have

$$\xi_n = 1 \mp 3r_n + O(r_n^2) \text{ as } n \to \infty,$$

and consequently,

$$\xi_2 \cdots \xi_n \sim c\left(n \log n \cdots \log_{K-2} n (\log_{K-1} n)^B\right)^{\mp 1},$$

$$\frac{\xi_2 \cdots \xi_n}{\sum_{i=1}^n \xi_2 \cdots \xi_i} \to 0, \text{ as } n \to \infty.$$

We now give the proof of Theorem 4. Suppose that $p_i = 1/3 \pm r_i, i \geq 2$. Then we have

$$P(M = n, D < \infty) = \mathcal{P}_2^n(1, n, +)(1 - \mathcal{P}_n(1, n+1, +))$$
$$\sim c \frac{1}{\sum_{s=2}^{n} \xi_2 \cdots \xi_{s-1}} \times \frac{\xi_2 \cdots \xi_n}{\sum_{s=2}^{n+1} \xi_2 \cdots \xi_{s-1}}, \text{ as } n \to \infty.$$

Furthermore,

$$\xi_2 \cdots \xi_n \sim c \left(n \log n \cdots \log_{K-2} n (\log_{K-1} n)^B \right)^{\mp 1}, \text{ as } n \to \infty.$$

With the above facts in hands, once again, the proof of Theorem 4 is just a step-by-step repetition of that of Theorem 1. □

Recurrence Criteria

Proposition

(i) For $K = 1$, if $q_i = \frac{2}{3} + r_i,\ i \geq 2$(or $p_i = \frac{1}{3} - r_i,\ i \geq 2$), then

$$B > 1 \Rightarrow Y \text{ is transient and } Y' \text{ is positive recurrent;}$$
$$B < -1 \Rightarrow Y' \text{ is transient and } Y \text{ is positive recurrent;}$$
$$B \in [-1, 1] \Rightarrow \text{both } Y \text{ and } Y' \text{ are null recurrent.}$$

(ii) For $K \geq 2$, if $q_i = \frac{2}{3} + r_i,\ i \geq 2$, then

$$B > 1 \Rightarrow Y \text{ is transient and } Y' \text{ is positive recurrent;}$$
$$B \leq 1 \Rightarrow \text{both } Y \text{ and } Y' \text{ are null recurrent.}$$

(iii) For $K \geq 2$, if $q_i = \frac{2}{3} - r_i,\ i \geq 2$, then

$$B > 1 \Rightarrow Y' \text{ is transient and } Y \text{ is positive recurrent;}$$
$$B \leq 1 \Rightarrow \text{both } Y \text{ and } Y' \text{ are null recurrent.}$$

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